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## Polynomials and pseudoconvexity for Riemann domains over $\mathbb{C}^n$

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**Abstract.** We prove that a Riemann domain  $(G, \pi)$  over  $\mathbb{C}^n$  is pseudoconvex if and only if for any continuous mapping  $\varphi : \overline{D} \times [0, \delta] \rightarrow \overline{\overline{G}}$  of the form  $(\overline{\pi} \circ \varphi)_j(w, t) = p_j(w) + a_j t$  ( $j = 1, 2, \dots, n$ ), where  $(\overline{\overline{G}}, \overline{\pi})$  is abstract closure of  $(G, \pi)$ ,  $D = \{w \in \mathbb{C} ; |w| < \varepsilon\}$ ,  $\varepsilon > 0$ ,  $\delta > 0$ ,  $a_j \in \mathbb{C}$  and  $p_j(w)$  is a polynomial of  $w$  of degree at most 2, with  $\varphi(\overline{D} \times (0, \delta]) \cup \varphi(\partial D \times \{0\}) \subset G$ , it follows that  $\varphi(\overline{D} \times [0, \delta]) \subset G$ .

### 1. Introduction

A pair  $(G, \pi)$  is called a *Riemann domain over  $\mathbb{C}^n$*  if  $G$  is a connected Hausdorff space and  $\pi : G \rightarrow \mathbb{C}^n$  is a local homeomorphism. There are several definition of pseudoconvexity for Riemann domains over  $\mathbb{C}^n$ . Among others, a Riemann domain  $(G, \pi)$  is *pseudoconvex* if it satisfies the continuity principle, that is, for any continuous mapping  $\varphi : \overline{D} \times [0, \delta] \rightarrow \overline{\overline{G}}$ , where  $D = \{w \in \mathbb{C} ; |w| < \varepsilon\}$ ,  $\varepsilon > 0$  and  $\delta > 0$ , such that  $(\overline{\pi} \circ \varphi)_j(w, t)$  is a holomorphic function of  $w$  in  $D$  for any  $t \in [0, \delta]$  and for any  $j \in \{1, 2, \dots, n\}$  with  $\varphi(\overline{D} \times (0, \delta]) \cup \varphi(\partial D \times \{0\}) \subset G$ , it follows that  $\varphi(\overline{D} \times [0, \delta]) \subset G$ . Here  $(\overline{\overline{G}}, \overline{\pi})$  is abstract closure of  $(G, \pi)$  (see Section 2). Yasuoka [3] proved that a domain  $\Omega$  in  $\mathbb{C}^n$  is pseudoconvex if and only if for any continuous mapping  $\varphi : \overline{D} \times [0, \delta] \rightarrow \mathbb{C}^n$ , where  $D = \{w \in \mathbb{C} ; |w| < \varepsilon\}$ ,  $\varepsilon > 0$  and  $\delta > 0$ , such that  $\varphi_j(w, t) = p_j(w) + a_j t$  ( $j = 1, 2, \dots, n$ ),  $a_j \in \mathbb{C}$ ,  $p_j(w)$  is a polynomial of  $w$  of degree at most 2 and  $(\partial \varphi_1 / \partial w, \partial \varphi_2 / \partial w, \dots, \partial \varphi_n / \partial w) \neq (0, 0, \dots, 0)$

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for any  $t \in [0, \delta]$  with  $\varphi(\overline{D} \times (0, \delta]) \cup \varphi(\partial D \times \{0\}) \subset \Omega$ , it follows that  $\varphi(\overline{D} \times [0, \delta]) \subset \Omega$ .

In this paper, we show that a Riemann domain  $(G, \pi)$  over  $\mathbb{C}^n$  is pseudoconvex if and only if for any continuous mapping  $\varphi : \overline{D} \times [0, \delta] \rightarrow \overline{\overline{G}}$ , where  $D = \{w \in \mathbb{C} ; |w| < \varepsilon\}$ ,  $\varepsilon > 0$  and  $\delta > 0$ , such that  $(\overline{\pi} \circ \varphi)_j(w, t) = p_j(w) + a_j t$  ( $j = 1, 2, \dots, n$ ),  $a_j \in \mathbb{C}$ ,  $p_j(w)$  is a polynomial of  $w$  of degree at most 2 with  $\varphi(\overline{D} \times (0, \delta]) \cup \varphi(\partial D \times \{0\}) \subset G$ , it follows that  $\varphi(\overline{D} \times [0, \delta]) \subset G$ .

## 2. Riemann domains and abstract boundary points

Let  $(G, \pi)$  be a Riemann domain over  $\mathbb{C}^n$  and let  $\overline{\partial}G$  be the set of all filter bases  $\alpha$  that satisfies the following four conditions.

- (1) There exists a point  $z_0 \in \mathbb{C}^n$  such that  $\lim \pi(\alpha) = z_0$ .
- (2) For any  $V \in \beta_c(z_0)$ , there exists exactly one connected component  $U$  of  $\pi^{-1}(V)$  such that  $U \in \alpha$ .
- (3) For any  $U \in \alpha$ , there exists  $V \in \beta_c(z_0)$  such that  $U$  is a connected component of  $\pi^{-1}(V)$ .
- (4)  $\alpha$  has no accumulation point in  $G$ .

Here  $\beta_c(z_0)$  is the set of all connected open neighborhoods of  $z_0$  in  $\mathbb{C}^n$ . The set  $\overline{\partial}G$  is called the *abstract boundary* of  $G$ . We put  $\overline{\overline{G}} = G \cup \overline{\partial}G$  and define  $\overline{\pi} : \overline{\overline{G}} \rightarrow \mathbb{C}^n$  by

$$\overline{\pi}(x) = \begin{cases} \pi(x) & (x \in G), \\ \lim \pi(x) & (x \in \overline{\partial}G). \end{cases}$$

The topology of  $\overline{\overline{G}}$  is as follows.

For every  $\alpha \in \overline{\partial}G$  and for every  $U \in \alpha$  we put

$$\widehat{U}_\alpha = U \cup \{\beta \in \overline{\partial}G ; \text{ there exists } W \in \beta \text{ such that } W \subset U\}.$$

Then  $\widehat{U}_\alpha$  is a fundamental neighborhood of  $\alpha$  and  $\overline{\pi}$  is continuous.  $(\overline{\overline{G}}, \overline{\pi})$  is said to be *abstract closure* of  $(G, \pi)$  (see Jarnicki–Pflug [1, p. 33]).

Let  $\mathcal{F}(G)$  be the set of all filter bases of  $G$  satisfying the above three conditions (1), (2) and (3). We define  $\sigma^G : \overline{\overline{G}} \rightarrow \mathcal{F}(G)$  by

$$\sigma^G(x) = \begin{cases} \alpha^x & (x \in G), \\ x & (x \in \overline{\partial}G), \end{cases}$$

where

$$\alpha^x = \{U_x ; \text{there exists } V \in \beta_c(\pi(x)) \text{ such that } U_x \text{ is a connected component of } \pi^{-1}(V) \text{ and } x \in U_x\}.$$

Then  $\sigma^G$  is well-defined. Moreover for every  $\alpha \in \mathcal{F}(G)$  and for every  $U \in \alpha$ , we put

$$U_\alpha = \{\beta \in \mathcal{F}(G) ; \text{there exists } W \in \beta \text{ such that } W \subset U\}.$$

Then the family  $\{U_\alpha ; U \in \alpha\}$  satisfies the axiom of fundamental system of neighborhoods. Therefore  $\mathcal{F}(G)$  is a topological space.

**Proposition 2.1.**  $\sigma^G$  is homeomorphic.

*Proof.* It is obvious that  $\alpha^x$  is a filter base that satisfies the above three conditions (1), (2), (3) and  $\lim \alpha^x = x$ . We show that  $\sigma^G$  is bijective. Obviously  $\sigma^G$  is injective. To see that  $\sigma^G$  is surjective, we put

$$\mathcal{F}_0(G) = \{\alpha \in \mathcal{F}(G) ; \alpha \text{ has an accumulation point in } G\}.$$

Then we have  $\mathcal{F}(G) = \mathcal{F}_0(G) \cup \overline{\partial}G$  and  $\mathcal{F}_0(G) \cap \overline{\partial}G = \emptyset$ . Let  $\alpha \in \mathcal{F}(G)$ . If  $\alpha \in \mathcal{F}_0(G)$ , we can put  $\lim \alpha = x \in G$  (see Jarnicki–Pflug [1, p. 30]) and see that  $\sigma^G(x) = \alpha^x = \alpha$ . If  $\alpha \in \overline{\partial}G$ , it is clear that  $\sigma^G(\alpha) = \alpha$ . Hence  $\sigma^G$  is surjective.

According to the definition of topology, we see that  $\sigma^G$  is homeomorphic.  $\square$

Therefore we can regard  $\overline{\overline{G}}$  as  $\mathcal{F}(G)$  by  $\sigma^G$ . Since  $\mathcal{F}(G)$  and  $\mathcal{F}_0(G)$  is useful, we sometimes use these symbols.

Next we consider a subdomain of a Riemann domain over  $\mathbb{C}^n$ . Let  $(G, \pi)$  be a Riemann domain over  $\mathbb{C}^n$  and let  $G_0$  be a subdomain of  $G$ . Then

$(G_0, \pi|_{G_0})$  is a Riemann domain over  $\mathbb{C}^n$ . We define the mapping which allows us to regard  $\overline{\overline{G_0}}$  as a subset of  $\overline{\overline{G}}$ . Let  $\mathcal{F}(G_0)$  be the set of all filter bases of  $G_0$  that satisfies the above three conditions (1), (2) and (3). For every  $\alpha \in \mathcal{F}(G_0)$ , we put

$$\begin{aligned} \hat{\alpha} = \{ & C_U \subset G ; \text{ there exist } U \in \alpha \text{ and } V \in \beta_c(\lim \pi(\alpha)) \\ & \text{such that } U \subset C_U \subset \pi^{-1}(V) \\ & \text{and } C_U \text{ is a connected component of } \pi^{-1}(V) \} \end{aligned}$$

and let  $\wedge : \mathcal{F}(G_0) \rightarrow \mathcal{F}(G)$ ,  $\alpha \mapsto \hat{\alpha}$ . Then the mapping  $\wedge$  is well-defined and continuous. We put  $\psi_{G_0} = (\sigma^G)^{-1} \circ \wedge \circ \sigma^{G_0}$ .

**Remark 2.1.** Let  $\alpha \in \overline{\overline{\partial G_0}}$ . If  $\alpha$  has an accumulation point  $x_\alpha$  in  $G$ , then  $\hat{\alpha} \in \mathcal{F}_0(G)$ . Especially,  $\lim \hat{\alpha} = \lim \alpha = x_\alpha$ . Since  $\pi$  is continuous, it follows that  $\pi(x_\alpha) = \pi(\lim \hat{\alpha}) = \pi(\lim \alpha) = \lim \pi(\alpha)$ . Then  $x_\alpha$  is unique, because  $G$  is a Hausdorff space. If  $\alpha$  has no accumulation point in  $G$ , then it is clear that  $\hat{\alpha} \in \overline{\overline{\partial G}}$ .

An open subset  $G_1$  of  $G$  is said to be *univalent* if  $\pi|_{G_1} : G_1 \rightarrow \pi(G_1)$  is homeomorphic.

**Lemma 2.1.** *Let  $(G, \pi)$  be a Riemann domain over  $\mathbb{C}^n$ , let  $G_0$  be a univalent subdomain of  $G$ . Assume that  $G_0$  satisfies the following condition.*

*For every  $z \in \partial\pi(G_0)$  and for every  $V \in \beta_c(z)$ , there exists  $V_0 \in \beta_c(z)$  such that  $V_0 \subset V$  and  $(\pi|_{G_0})^{-1}(V_0)$  is connected.*

*Then the following two statements hold.*

- (1)  $\overline{\overline{\pi|_{G_0}}} : \overline{\overline{G_0}} \rightarrow \overline{\overline{\pi(G_0)}}$  is homeomorphic.
- (2)  $\psi_{G_0} : \overline{\overline{G_0}} \rightarrow \psi_{G_0}(\overline{\overline{G_0}})$  is homeomorphic.

*Proof.* (1) We define  $f : \partial\pi(G_0) \rightarrow \overline{\overline{\partial G_0}}$ ,  $z \mapsto \alpha_z$ . Here  $\alpha_z$  is an abstract boundary point of  $G_0$  with  $\lim \pi(\alpha_z) = z$ . Then  $f$  is well-defined. In fact, we put

$$\begin{aligned} \alpha_z = \{ & C ; \text{ there exist } V \in \beta_c(z) \text{ and } \{x_\nu\}_{\nu \in \mathbb{N}} \subset G \text{ such that} \\ & C \text{ is a connected component of } \pi^{-1}(V) \cap G_0 \text{ and} \\ & \text{almost all } x_\nu \text{ lie in } C \text{ and } \lim_{\nu \rightarrow +\infty} \pi(x_\nu) = z \}. \end{aligned}$$

We shall show  $\alpha_z \in \overline{\partial}G_0$ ,  $\lim \pi(\alpha_z) = z$  and  $\alpha_z \neq \emptyset$ . It is clear that  $\emptyset \notin \alpha_z$  and  $\alpha_z \neq \emptyset$ .

Let  $C_1$  and  $C_2$  be elements of  $\alpha_z$ . Then there exist  $\{x_\nu^{(1)}\}_{\nu \in \mathbb{N}} \subset G$  and  $\{x_\nu^{(2)}\}_{\nu \in \mathbb{N}} \subset G$  such that  $\lim_{\nu \rightarrow +\infty} \pi(x_\nu^{(1)}) = z$  and  $\lim_{\nu \rightarrow +\infty} \pi(x_\nu^{(2)}) = z$ . And there exist  $V_1 \in \beta_c(z)$ ,  $V_2 \in \beta_c(z)$ ,  $\nu_1 \in \mathbb{N}$  and  $\nu_2 \in \mathbb{N}$  such that  $\{x_\nu^{(1)}\}_{\nu \geq \nu_1} \subset C_1 \subset \pi^{-1}(V_1) \cap G_0$  and  $\{x_\nu^{(2)}\}_{\nu \geq \nu_2} \subset C_2 \subset \pi^{-1}(V_2) \cap G_0$ , where  $C_1$  is a connected component of  $\pi^{-1}(V_1) \cap G_0$  and  $C_2$  is a connected component of  $\pi^{-1}(V_2) \cap G_0$ . Since  $V_1 \cap V_2$  is an open neighborhood of  $z$ , there is  $V_0 \in \beta_c(z)$  such that  $\pi^{-1}(V_0) \cap G_0$  is connected,  $V_0 \subset V_1 \cap V_2$  and there is  $N \in \mathbb{N}$  such that for every  $\nu > N$ , we get  $x_\nu^{(1)}, x_\nu^{(2)} \in \pi^{-1}(V_0)$ . We obtain  $(\pi^{-1}(V_0) \cap G_0) \cap C_1 \neq \emptyset$  and  $(\pi^{-1}(V_0) \cap G_0) \cap C_2 \neq \emptyset$ . Hence  $\pi^{-1}(V_0) \cap G_0 \subset C_1 \cap C_2$  and  $\pi^{-1}(V_0) \cap G_0 \in \alpha_z$ . Therefore  $\alpha_z$  is a filter base of  $G_1$ .

For any  $V \in \beta_c(z)$ , there is  $V_0 \in \beta_c(z)$  such that  $V_0 \subset V$  and  $\pi^{-1}(V_0) \cap G_0$  is connected. Then we obtain  $\pi^{-1}(V_0) \cap G_0 \in \alpha_z$  and  $\pi(\pi^{-1}(V_0) \cap G_0) \subset V_0 \cap \pi(G_0) \subset V_0 \subset V$ . Hence  $\lim \pi_1(\alpha_z) = z$ .

We show that for any  $V \in \beta_c(z)$ , there exists exactly one connected component  $C$  of  $\pi^{-1}(V) \cap G_0$  such that  $C \in \alpha_z$ . For any  $V \in \beta_c(z)$ , let  $C_1$  and  $C_2$  be connected components of  $\pi^{-1}(V) \cap G_0$  that satisfy the following condition. There exist  $\{x_\nu^{(i)}\}_{\nu \in \mathbb{N}} \subset G$  ( $i = 1, 2$ ) and  $\nu_i \in \mathbb{N}$  ( $i = 1, 2$ ) such that  $\{x_\nu^{(i)}\}_{\nu \geq \nu_i} \subset C_i$  ( $i = 1, 2$ ) and  $\lim_{\nu \rightarrow +\infty} \pi_1(x_\nu^{(i)}) = z$  ( $i = 1, 2$ ). By the assumption, there exists  $V_0 \in \beta_c(z)$  such that  $V_0 \subset V$  and  $\pi^{-1}(V_0) \cap G_0$  is connected. Now  $V_0$  contains almost all  $\{\pi(x_\nu^{(1)})\}_{\nu \in \mathbb{N}}$  and  $\{\pi(x_\nu^{(2)})\}_{\nu \in \mathbb{N}}$ . Thus  $\pi^{-1}(V_0) \cap G_0 \cap C_1 \neq \emptyset$  and  $\pi^{-1}(V_0) \cap G_0 \cap C_2 \neq \emptyset$ . Since  $C_1$  and  $C_2$  are connected components of  $\pi^{-1}(V_0) \cap G_0$ , we have  $\pi^{-1}(V_0) \cap G_0 \subset C_1$  and  $\pi^{-1}(V_0) \cap G_0 \subset C_2$ . It follows that  $C_1 = C_2$ .

Obviously  $\alpha_z$  satisfies that for any  $U \in \alpha_z$ , there exists  $V \in \beta_c(z)$  such that  $U$  is a connected component of  $\pi^{-1}(V)$ . It is clear that  $\alpha_z$  has no accumulation point. Therefore  $\alpha_z$  is an abstract boundary point with  $\lim \pi(\alpha_z) = z$ .

Then this  $\alpha_z$  is unique. In fact, suppose that  $\alpha'$  is an abstract boundary point of  $G_0$  with  $\lim \pi(\alpha') = z$ . Assume that  $\alpha' \neq \alpha_z$ . Then there exist  $U' \in \alpha'$  and  $U \in \alpha_z$  such that  $U \cap U' = \emptyset$ . Moreover there exist  $V \in \beta_c(z)$  and  $V' \in \beta_c(z)$  such that  $U$  is a connected component of  $\pi^{-1}(V) \cap G_0$  and

$U'$  is a connected component of  $\pi^{-1}(V') \cap G_0$ . Now  $V \cap V'$  is an open neighborhood of  $z$ . Thus there is  $V_0 \in \beta_c(z)$  such that  $V_0 \subset V \cap V'$  and  $\pi^{-1}(V_0) \cap G_0$  is connected by the assumption of  $G_0$ . Then it follows that  $U \cap \pi^{-1}(V_0) \subset \pi^{-1}(V_0) \cap G_0$  and  $U' \cap \pi^{-1}(V_0) \subset \pi^{-1}(V_0) \cap G_0$ .

Let  $\{x_\nu\}_{\nu \in \mathbb{N}}$  be determined by  $U$  and let  $\{x'_\nu\}_{\nu \in \mathbb{N}}$  be determined by  $U'$ . Then there is  $N \in \mathbb{N}$  such that for every  $\nu > N$ , we have  $\pi(x_\nu) \in V_0$  and  $\pi(x'_\nu) \in V_0$ . Hence  $U \cap \pi^{-1}(V_0) \neq \emptyset$  and  $U' \cap \pi^{-1}(V_0) \neq \emptyset$ . Therefore we have  $U \cap \pi^{-1}(V_0) \supset \pi^{-1}(V_0) \cap G_0$  and  $U' \cap \pi^{-1}(V_0) \supset \pi^{-1}(V_0) \cap G_0$ . It follows that  $U \cap \pi^{-1}(V_0) = \pi^{-1}(V_0) \cap G_0$  and  $U' \cap \pi^{-1}(V_0) = \pi^{-1}(V_0) \cap G_0$ . This is a contradiction. Hence  $f$  is well-defined. It is easy to see that  $f$  is bijective.

Define  $F : \overline{\pi(G_0)} \rightarrow \overline{G_0}$  as  $F|_{\partial\pi(G_0)} = f$  and  $F|_{\pi(G_0)} = (\pi|_{G_0})^{-1}$ . Then  $F$  is homeomorphic. In fact, we put  $z \in \partial\pi(G_0)$  and  $f(z) = \alpha_z$ . Let  $\widehat{U}_{\alpha_z} = U \cup \{\beta \in \overline{\partial G_0} ; \text{there exists } W \in \beta \text{ such that } W \subset U\}$  be a neighborhood of  $\alpha_z$ . Then there exists  $V \in \beta_c(z)$  such that  $U$  is a connected component of  $\pi^{-1}(V) \cap G_0$ . By the assumption of  $G_0$ , there is  $V_0 \in \beta_c(z)$  such that  $V_0 \subset V$  and  $\pi^{-1}(V_0) \cap G_0$  is connected. We shall prove that  $F(V_0 \cap \overline{\pi(G_0)}) \subset \widehat{U}_{\alpha_z}$ . It is clear that  $F(V_0 \cap \pi(G_0)) = \pi^{-1}(V_0 \cap \pi(G_0)) = \pi^{-1}(V_0) \cap G_0 \subset U$ . Hence we only have to show that  $F(V_0 \cap \partial\pi(G_0)) \subset \{\beta \in \overline{\partial G_0} ; \text{there exists } W \in \beta \text{ such that } W \subset U\}$ . Let  $z_0 \in V_0 \cap \partial\pi(G_0)$  and  $F(z_0) = \alpha_0$ , since  $V_0 \in \beta_c(z_0)$ , there is  $C_0 \in \alpha_0$  such that  $C_0 \subset \pi^{-1}(V_0) \cap G_0$ . Since  $\pi^{-1}(V_0) \cap G_1$  is connected, then  $C_0 = \pi^{-1}(V_0) \cap G_1 \subset U$ . Thus  $\alpha_0 \in \{\beta \in \overline{\partial G_0} ; \text{there exists } W \in \beta \text{ such that } W \subset U\}$ . Consequently,  $F$  is continuous. Since  $F^{-1} = \overline{\pi_1}$ ,  $F$  is homeomorphic. Therefore  $\overline{\pi|_{G_0}}$  is homeomorphic.

Next we shall show (2). Put  $\psi_{G_0} = \psi$ . Then  $\text{id}_{\partial\pi(G_0)} = \overline{\pi} \circ \psi \circ \overline{\pi|_{G_0}}^{-1} \Big|_{\partial\pi(G_0)}$  and  $\text{id}_{\psi(\overline{\partial G_0})} = \psi \circ \overline{\pi|_{G_0}}^{-1} \circ \overline{\pi} \Big|_{\psi(\overline{\partial G_0})}$  hold.

In fact, let  $z \in \partial\pi(G_0)$  and let  $\overline{\pi|_{G_0}}^{-1}(z) = \alpha$ , where  $\alpha$  is an abstract boundary point of  $G_0$  with  $\lim \pi(\alpha) = z$ .

**Case 1 :**  $\widehat{\alpha} \in \mathcal{F}_0(G)$ .

Then  $\psi \circ \overline{\pi|_{G_0}}^{-1}(z) = \psi(\alpha) = (\sigma^G)^{-1}(\widehat{\alpha}) = \lim \widehat{\alpha} = \lim \alpha \in G$ , it follows that  $\overline{\pi} \circ \psi \circ \overline{\pi|_{G_0}}^{-1}(z) = \overline{\pi}(\lim \widehat{\alpha}) = \overline{\pi}(\lim \alpha) = \pi(\lim \alpha) = \lim \pi(\alpha) = z$ .

**Case 2 :**  $\widehat{\alpha} \in \overline{\partial G}$ .

Then  $\bar{\pi}(\hat{\alpha}) = \lim \pi(\hat{\alpha}) = \lim \pi(\alpha) = z$ .

Thus we obtain  $\text{id}_{\partial\pi(G_0)} = \bar{\pi} \circ \psi \circ \overline{\pi|_{G_0}}^{-1} \Big|_{\partial\pi(G_0)}$ .

It remains to show that  $\text{id}_{\psi(\bar{\partial}G_0)} = \psi \circ \overline{\pi|_{G_0}}^{-1} \circ \bar{\pi}|_{\psi(\bar{\partial}G_0)}$ . Let  $x \in \psi(\bar{\partial}G_0)$ .

**Case 1 :**  $x \in G$ .

Then  $\psi \circ \overline{\pi|_{G_0}}^{-1} \circ \bar{\pi}(x) = \psi \circ \overline{\pi|_{G_0}}^{-1} \circ \pi(x) = \psi(\alpha)$ , where  $\alpha$  is an abstract boundary point of  $G_0$  with  $\lim \pi(\alpha) = \pi(x)$ . There is  $\beta \in \bar{\partial}G_0$  such that  $\lim \hat{\beta} = \lim \beta = x \in G$ . Thus  $\lim \pi(\beta) = \pi(x)$ . It follows from bijectivity of  $\overline{\pi|_{G_0}}^{-1}$  that  $\beta = \alpha$ . Therefore,  $\psi(\alpha) = \lim \alpha = \lim \beta = x$ .

**Case 2 :**  $x \in \bar{\partial}G$ .

Then there exists  $\beta \in \bar{\partial}G_0$  such that  $\psi(\beta) = \hat{\beta} = x$ . We get  $\psi \circ \overline{\pi|_{G_0}}^{-1} \circ \bar{\pi}(x) = \psi \circ \overline{\pi|_{G_0}}^{-1} \circ \bar{\pi}(\hat{\beta}) = \psi \circ \overline{\pi|_{G_0}}^{-1} (\lim \pi(\hat{\beta})) = \psi(\alpha)$ .

Here  $\alpha$  is an abstract boundary point of  $G_0$  with  $\lim \pi(\alpha) = \lim \pi(\hat{\beta}) = \lim \pi(\beta)$ . It follows from bijectivity of  $\overline{\pi|_{G_0}}^{-1}$  that  $\beta = \alpha$ . Thus  $\psi(\alpha) = \psi(\beta) = \hat{\beta} = x$ . It follows that  $\text{id}_{\psi(\bar{\partial}G_0)} = \psi \circ \overline{\pi|_{G_0}}^{-1} \circ \bar{\pi}|_{\psi(\bar{\partial}G_0)}$ .

We obtain that  $\psi|_{\psi(\bar{\partial}G_0)}^{-1} = \overline{\pi|_{G_0}}^{-1} \circ \bar{\pi}|_{\psi(\bar{\partial}G_0)}$ . It follows that  $\psi : \bar{G}_0 \rightarrow \psi(\bar{G}_0)$  is homeomorphic.  $\square$

Let  $G_0 \subset G$  be an open univalent neighborhood of  $x \in G$ , let  $|\cdot|$  be the maximum norm in  $\mathbb{C}^n$  and let  $r \in (0, +\infty]$ . If

$$\pi(G_0) = P(\pi(x), r) = \{z \in \mathbb{C}^n ; |\pi(x) - z| < r\},$$

then  $G_0$  is called a *polydisk* with radius  $r$  and center  $x$  and is denoted by  $\hat{P}(x, r)$ . We define

$$\delta_G(x) = \sup\{r \in (0, +\infty] ; \hat{P}(x, r) \text{ exists}\},$$

which is called the *boundary distance function*. The set  $\hat{P}(x, \delta_G(x))$  is called the *maximal polydisk* with center  $x$ .

**Corollary 2.1.** *Let  $(G, \pi)$  be a Riemann domain over  $\mathbb{C}^n$  and let  $\hat{P}(x, \delta_G(x))$  be a maximal polydisk. Then both  $\overline{\pi|_{\hat{P}(x, \delta_G(x))}} : \overline{\hat{P}(x, \delta_G(x))} \rightarrow \overline{P(x, \delta_G(x))}$  and  $\psi|_{\hat{P}(x, \delta_G(x))} : \overline{\hat{P}(x, \delta_G(x))} \rightarrow \overline{\psi(\hat{P}(x, \delta_G(x)))}$  are homeomorphic.*

### 3. $O_2$ -pseudoconvex domains and pseudoconvex domains

After defining the  $O_m$ -pseudoconvexity ( $m \in \mathbb{N}$ ), we show that the pseudoconvexity is equivalent to the  $O_2$ -pseudoconvexity. Let  $(G, \pi)$  be a Riemann domain over  $\mathbb{C}^n$ .

**Definition 3.1.** Let  $m \in \mathbb{N}$  and let  $\varphi : \overline{D} \times [0, \delta] \rightarrow \overline{G}$  be a continuous map, where  $D = \{w \in \mathbb{C} ; |w| < \varepsilon\}$ ,  $\varepsilon > 0$  and  $\delta > 0$ . If  $(\overline{\pi} \circ \varphi)_j(w, t) = p_j(w) + a_j t$  ( $j = 1, 2, \dots, n$ ),  $a_j \in \mathbb{C}$ ,  $p_j(w)$  is a polynomial of  $w$  of degree at most  $m$ , then  $\varphi$  is called a *family of analytic disks of degree  $m$* .

**Definition 3.2.** We say that  $G$  is  $O_m$ -pseudoconvex if for any family  $\varphi$  of analytic disks of degree  $m$  with  $\varphi(\overline{D} \times (0, \delta]) \cup \varphi(\partial D \times \{0\}) \subset G$ , we have  $\varphi(\overline{D} \times [0, \delta]) \subset G$ .

**Remark 3.1.** The  $O_m$ -pseudoconvexity is invariant under affine transformations.

For any  $a \in \mathbb{C}$  and for any  $\varepsilon \in (0, +\infty]$ , the set  $\{z \in \mathbb{C} ; |z - a| < \varepsilon\}$  is denoted by  $D(a, \varepsilon)$ .

**Lemma 3.1** (Yasuoka [3, Lemma 1]). *Let  $\Omega \subset \mathbb{C}$  be a domain and let  $f : \Omega \rightarrow [-\infty, +\infty)$  be an upper semi-continuous function. If  $f$  is not subharmonic on  $\Omega$ , then there exist  $a \in \Omega$ ,  $D = D(a, \varepsilon) \Subset \Omega$ ,  $h \in C^\infty(\overline{D})$  and  $C > 0$  such that*

$$\begin{cases} h_{z\bar{z}}(z) = -C & \text{for } z \in D, \\ h(a) = f(a), \\ h(z) \geq f(z) & \text{for } z \in D. \end{cases}$$

**Theorem 3.1.** *Let  $(G, \pi)$  be a Riemann domain over  $\mathbb{C}^n$ . Then the following two statements are equivalent.*

- (1)  $(G, \pi)$  is pseudoconvex.
- (2)  $(G, \pi)$  is  $O_2$ -pseudoconvex.

*Proof.* The implication (1) $\Rightarrow$ (2) is trivial. We show another implication. We can assume that  $G$  and  $\mathbb{C}^n$  are not homeomorphic. Seeking a contradiction, suppose that  $(G, \pi)$  is not pseudoconvex. Then  $-\log \delta_G(x)$  is not



plurisubharmonic on  $G$  (see Jarnicki–Pflug[1, p. 143]). By an affine transformation which conserves the distance, we can assume that  $-\log \delta_G(\pi_x^{-1}(w\xi_0))$  is not subharmonic on  $D(0, \varepsilon) \subset \{w \in \mathbb{C} ; |w\xi_0| < \delta_G(x)\}$ . Here  $x \in G$ ,  $\varepsilon > 0$ ,  $\pi_x = \pi|_{\hat{P}(x, \delta_G(x))}$  and  $\xi_0 = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ . It follows from Lemma 3.1 that there exist  $a_0 \in D(0, \varepsilon)$ ,  $D_0 = D(a_0, \varepsilon_0) \Subset D(0, \varepsilon)$ ,  $h \in C^\infty(\overline{D_0})$  and  $C > 0$  such that

$$\begin{cases} -h_{z\bar{z}}(w) = C & \text{for } w \in D_0, \\ -h(a_0) = \log \delta_G(\pi_x^{-1}(a_0\xi_0)), \\ -h(w) \leq \log \delta_G(\pi_x^{-1}(w\xi_0)) & \text{for } w \in D_0. \end{cases}$$

By translation, we may let  $a_0 = 0$ . Put

$$\begin{aligned} \hat{P}(w) &= \hat{P}(\pi_x^{-1}(w\xi_0), \delta_G(\pi_x^{-1}(w\xi_0))) \text{ and} \\ P(w) &= P(w\xi_0, \delta_G(\pi_x^{-1}(w\xi_0))). \end{aligned}$$

Then we consider the maximal polydisk  $\hat{P}(0)$ . By Corollary 2.1,  $\partial P(0)$  and  $\overline{\partial} \hat{P}(0)$  are homeomorphic. Moreover  $\overline{\partial} G \cap \psi_{\hat{P}(0)}(\overline{\partial} \hat{P}(0)) \neq \emptyset$ . Then there is  $u \in \overline{\partial} G \cap \psi_{\hat{P}(0)}(\overline{\partial} \hat{P}(0))$  such that  $\overline{\pi}(u) \in \partial P(0)$ . We can assume that there exist  $z_{k+1}^{(0)}, \dots, z_n^{(0)} \in D(0, \delta_G(\pi_x^{-1}(0)))$  such that

$$\overline{\pi}(u) = (\delta_G(\pi_x^{-1}(0)), \dots, \delta_G(\pi_x^{-1}(0)), z_{k+1}^{(0)}, \dots, z_n^{(0)}) \in \partial P(0).$$

Define  $h_1(w) = -h(w) - C|w|^2$ . Then  $h_1$  is harmonic on  $D_0$ . Since  $D_0$  is simply connected, there exists exactly one conjugate harmonic function  $h_2$  on  $D_0$  with  $h_2(0) = 0$ .

Let  $p(w) + (\text{terms of order } \geq 3)$  be the power series expansion of the holomorphic function  $\exp(h_1(w) + ih_2(w))$  at  $w = 0$ .

For any  $\delta \in (0, +\infty]$ , we define the family  $\chi : \overline{D_0} \times [0, \delta] \rightarrow \mathbb{C}^n$  of analytic

disks of degree 2 in  $\mathbb{C}^n$  by

$$\chi(w, t) = \begin{cases} \chi_1(w, t) = p(w) - t + w\xi_1, \\ \chi_2(w, t) = p(w) - t + w\xi_2, \\ \vdots \\ \chi_k(w, t) = p(w) - t + w\xi_k, \\ \chi_{k+1}(w, t) = z_{k+1}^{(0)} + w\xi_{k+1}, \\ \vdots \\ \chi_n(w, t) = z_n^{(0)} + w\xi_n. \end{cases}$$

We can choose  $D_0$  so that

$$|p(w) - t| \leq |\exp(h_1(w) + ih_2(w)) - t| + L_1|w|^3$$

for all  $(w, t) \in D_0 \times [0, \delta]$ , where  $L_1$  is a positive constant.

Moreover we can assume that  $0 < \delta_G(\pi_x^{-1}(0)) < 1$  by Remark 3.1. Let  $\varepsilon_0$  and  $\delta$  be sufficiently small. Then we obtain

$$|\exp(h_1(w) + ih_2(w) - t)| \geq |\exp(h_1(w) + ih_2(w)) - t|$$

for all  $(w, t) \in D_0 \times [0, \delta]$ . Thus

$$|p(w) - t| \leq \exp(h_1(w) - t) + L_1|w|^3 \quad (1)$$

for all  $(w, t) \in D_0 \times [0, \delta]$ . Since  $h_1(0) > 0$ , we can easily prove that

$$\log |p(w) - t| \leq h_1(w) - t + L_2|w|^3$$

for all  $(w, t) \in D_0 \times [0, \delta]$ , where  $L_2$  is a positive constant.

For any  $\varepsilon_1 \in (0, \min\{\varepsilon_0, \frac{C}{L_2}\})$  and put  $D = D(0, \varepsilon_1)$ . Then we have

$$h_1(w) - t + L_2|w|^3 \leq h_1(w) + C|w|^2 - t = -h(w) - t \quad (2)$$

for all  $(w, t) \in \overline{D} \times [0, \delta]$ . Then we consider  $|\chi(w, t) - w\xi_0|$ .

**Case 1 :**  $|\chi(w, t) - w\xi_0| = |p(w) - t|$ .

Inequality (2) implies

$$\begin{aligned} \log |\chi(w, t) - w\xi_0| &= \log |p(w) - t| \\ &\leq -h(w) - t \leq \log \delta_G(\pi_x^{-1}(w\xi_0)) - t \end{aligned}$$

for all  $(w, t) \in \overline{D} \times [0, \delta]$ . Therefore we have

$$|\chi(w, t) - w\xi_0| < \delta_G(\pi_x^{-1}(w\xi_0))$$

for all  $(w, t) \in \overline{D} \times (0, \delta]$ .

**Case 2 :** There is  $l \in \{k+1, k+2, \dots, n\}$  such that  $|\chi(w, t) - w\xi_0| = |z_l^{(0)}|$ .

We can choose  $\overline{D}$  so that

$$|z_l^{(0)}| < \delta_G(\pi_x^{-1}(w\xi_0))$$

for all  $w \in \overline{D}$  by continuity of  $\delta_G(\pi_x^{-1}(w\xi_0))$ .

Thus Case 1 and Case 2 imply

$$|\chi(w, t) - w\xi_0| < \delta_G(\pi_x^{-1}(w\xi_0)) \quad (3)$$

for all  $(w, t) \in \overline{D} \times (0, \delta]$ . By inequality (2), for every  $w \in \partial D$ , we get

$$h_1(w) + L_2|w|^3 < h_1(w) + C|w|^2 = -h(w).$$

Hence

$$\log |p(w)| \leq h_1(w) + L_2|w|^3 < -h(w) \leq \log \delta_G(\pi_x^{-1}(w\xi_0))$$

for all  $w \in \partial D$ . Consequently, we have  $|p(w)| < \delta_G(\pi_x^{-1}(w\xi_0))$  for any  $w \in \partial D$ . It follows that

$$|\chi(w, 0) - w\xi_0| < \delta_G(\pi_x^{-1}(w\xi_0)) \quad (4)$$

for all  $w \in \partial D$ . We made preparations to define the family of analytic disks of degree 2 of  $\overline{G}$ . Put  $G_0 = \bigcup_{w \in \overline{D}} \widehat{P}(w)$  and  $\pi_{G_0} = \pi|_{G_0}$ . Then  $G_0$  is connected and  $\pi_{G_0}$  is homeomorphic (cf. Narasimhan [2, p. 107]). Define  $J : \overline{D} \times (0, \delta] \rightarrow G_0 \subset \overline{G}$  by  $J(w, t) = \pi_w^{-1} \circ \chi(w, t)$ . We shall show that  $J$  is continuous. For any  $(w, t) \in \overline{D} \times (0, \delta]$ , let  $U(J(w, t))$  be an open neighborhood of  $J(w, t)$ . Then we can assume that  $U(J(w, t)) \subset \widehat{P}(w)$ . Since  $\pi_w$  is open,  $\pi_w(U(J(w, t)))$  is an open neighborhood of  $\chi(w, t)$ . Now  $\chi(w, t)$  is continuous. Therefore there is a neighborhood  $D' \times T'$  of  $(w, t)$  such that  $\chi(D' \times T') \subset \pi_w(U(J(w, t))) \subset P(w)$ . Then we have  $\pi_w^{-1}(\chi(D' \times T')) \subset U(J(w, t))$ .

$T')) \subset U(J(w, t)) \subset \widehat{P}(w)$ . Moreover there is an open neighborhood  $D''$  of  $w$  such that

$$|w\xi_0 - w'\xi_0| < \delta_G(\pi_x^{-1}(w\xi_0))$$

for all  $w' \in D''$ . Therefore for every  $w' \in D''$ , we get  $P(w) \cap P(w') \neq \emptyset$  and  $\pi_w^{-1} = \pi_{w'}^{-1}$  on  $P(w) \cap P(w')$ . Put  $D' \cap D'' = D'''$ . Then  $D'''$  is an open neighborhood of  $w$  and we obtain  $\chi(D''' \times T') \subset P(w)$ . Therefore  $\pi_{w'}^{-1}(\chi(D''' \times T')) \subset \pi_w^{-1}(\chi(D''' \times T')) \subset U(J(w, t))$  for any  $w' \in D'''$ . It follows that  $J(D''' \times T') \subset U(J(w, t))$ . This means that  $J$  is continuous. Next we extend  $J$ . we define  $\overline{\overline{J}} : \overline{\overline{D}} \times [0, \delta] \rightarrow \overline{\overline{G}}$  by

$$\overline{\overline{J}}(w, t) = \begin{cases} J(w, t) & (t \neq 0), \\ \lim_{\nu \rightarrow +\infty} J(w_\nu, t_\nu) & (t = 0), \end{cases}$$

where  $\{(w_\nu, t_\nu)\}_{\nu \in \mathbb{N}}$  satisfies  $(w_\nu, t_\nu) \rightarrow (w, 0)$  ( $\nu \rightarrow +\infty$ ) and  $t_\nu \neq 0$  for any  $\nu \in \mathbb{N}$ . Then  $\overline{\overline{J}}$  is well-defined and continuous. In fact, first we shall show that the sequence  $\{x_\nu\}_{\nu \in \mathbb{N}} = \{J(w_\nu, t_\nu)\}_{\nu \in \mathbb{N}} = \{\pi_{w_\nu}^{-1} \circ \chi(w_\nu, t_\nu)\}_{\nu \in \mathbb{N}}$  has a limit point.

**Case 1 :**  $\chi(w, 0) \in P(w)$ .

We have  $\pi_w^{-1} \circ \chi(w, 0) \in \widehat{P}(w) \subset G$ . Then we show that  $\lim_{\nu \rightarrow +\infty} \pi_{w_\nu}^{-1} \circ \chi(w_\nu, t_\nu) = \pi_w^{-1} \circ \chi(w, 0)$ . Let  $U = U(\pi_w^{-1} \circ \chi(w, 0)) \subset \widehat{P}(w)$  an open neighborhood of  $\pi_w^{-1} \circ \chi(w, 0)$ , then  $\pi(U)$  is an open neighborhood of  $\chi(w, 0)$ . Since  $\chi(w, t)$  is continuous, there exists  $N \in \mathbb{N}$  such that for every  $\nu > N$ , we get  $\chi(w_\nu, t_\nu) \in \pi(U) \subset P(w)$ . Thus for any  $\nu > N$ , we have  $P(w) \cap P(w_\nu) \neq \emptyset$  and  $\pi_w^{-1} = \pi_{w_\nu}^{-1}$  on  $P(w) \cap P(w_\nu)$ . Therefore  $\pi_w^{-1} \circ \chi(w_\nu, t_\nu) = \pi_{w_\nu}^{-1} \circ \chi(w_\nu, t_\nu) \in U$  for all  $\nu > N$ . We get  $\lim_{\nu \rightarrow +\infty} \pi_{w_\nu}^{-1} \circ \chi(w_\nu, t_\nu) = \pi_w^{-1} \circ \chi(w, 0)$ .

**Case 2 :**  $\chi(w, 0) \in \partial P(w)$ .

We get  $(\overline{\overline{\pi}}_w)^{-1} \circ \chi(w, 0) \in \overline{\overline{\partial P}}(w)$  by Corollary 1. Then  $\wedge \circ (\overline{\overline{\pi}}_w)^{-1} \circ \chi(w, 0) \in \mathcal{F}(G)$ .

**Case 2.1 :**  $\wedge \circ (\overline{\overline{\pi}}_w)^{-1} \circ \chi(w, 0) = \alpha \in \mathcal{F}_0(G)$ .

We can put  $\lim \alpha = x$ . Then  $\pi(x) = \chi(w, 0)$ . We shall show  $\lim_{\nu \rightarrow +\infty} \pi_{w_\nu}^{-1} \circ \chi(w_\nu, t_\nu) = x$ . First we show for any  $V \in \beta_c(\chi(w, 0))$ , there exists  $N \in \mathbb{N}$  such that for any  $\nu > N$ , we get  $x_\nu \in C_U \subset \pi^{-1}(V)$ . Here  $C_U$  is an

element of  $\alpha$  with  $U \subset C_U$  and  $U$  is connected component of  $\pi_w^{-1}(V)$  with  $U \in (\bar{\pi}_w)^{-1} \circ \chi(w, 0)$ . Assume that there exists  $V \in \beta_c(\chi(w, 0))$  such that for any  $N \in \mathbb{N}$ , there exists  $\nu > N$  such that  $x_\nu \notin C_U \subset \pi^{-1}(V)$ . It leads to a contradiction. We can assume that  $V$  is sufficiently small. Then we obtain a subsequence  $\{x_{\nu(j)}\}_{j \in \mathbb{N}} \subset \{x_\nu\}_{\nu \in \mathbb{N}}$  with  $x_{\nu(j)} \notin C_U$  for every  $j \in \mathbb{N}$ . Put  $\pi(x_{\nu(j)}) = \chi(w_{\nu(j)}, t_{\nu(j)}) = \zeta_j$ , then  $\zeta_j \in P(w_{\nu(j)})$  for every  $j \in \mathbb{N}$  and there is  $N \in \mathbb{N}$  such that for every  $j > N$ , we have  $\zeta_j \in V$ . Moreover we obtain  $P(w_{\nu(j)}) \cap P(w) = \emptyset$  for any  $j > N$ . In fact, suppose that there exists  $j_0 \in \mathbb{N}$  such that  $P(w_{\nu(j_0)}) \cap P(w) \neq \emptyset$ . Since  $V \cap P(w_{\nu(j)}) \neq \emptyset$  and  $V \cap P(w) \neq \emptyset$ , we can take a sufficiently small polydisk  $V$  such that  $V \cap (P(w_{\nu(j)}) \cap P(w))$  is connected. Therefore  $\pi_w^{-1}(V) = U \subset (\pi|_{\hat{P}(w) \cup \hat{P}(w_{\nu(j_0)})})^{-1}(V) \subset C_U \subset \pi^{-1}(V)$ . Thus we get  $x_{\nu(j_0)} \in C_U$ . This is a contradiction. It follows that  $P(w_{\nu(j)}) \cap P(w) = \emptyset$  for any  $j > N$ . However since  $w_{\nu(j)}\xi_0 \rightarrow w\xi_0$  ( $j \rightarrow +\infty$ ), this is also a contradiction. Therefore  $\lim_{\nu \rightarrow +\infty} x_\nu = x$ .

**Case 2.2 :**  $\wedge \circ (\bar{\pi}_w)^{-1} \circ \chi(w, 0) = \alpha \in \bar{\partial}G$ .

By the same as above, we get  $\lim_{\nu \rightarrow +\infty} x_\nu = \alpha$ . Then the limit value is independent of the choice of a sequence  $\{(w_\nu, t_\nu)\}_{\nu \in \mathbb{N}}$  with  $(w_\nu, t_\nu) \rightarrow (w, 0)$  ( $\nu \rightarrow +\infty$ ) and  $t_\nu \neq 0$  for any  $\nu \in \mathbb{N}$ . Hence  $\bar{J}$  is well-defined.

Next we show that  $\bar{J}$  is continuous. We consider a sequence  $\{(w_\nu, t_\nu)\}_{\nu \in \mathbb{N}}$  which satisfies  $t_i = 0$  for some  $i \in \mathbb{N}$ . In this case, since we need a sequence with  $(w_\nu, t_\nu) \rightarrow (w_i, 0)$  ( $\nu \rightarrow +\infty$ ) and  $t_\nu \neq 0$  for any  $\nu \in \mathbb{N}$ , we define a double sequence as follows. When  $i_1 \in \mathbb{N}$  satisfies  $(w_{i_1}, t_{i_1}) = (w_{i_1}, 0)$ , we take a sequence  $\{t_{i_1, j}\}_{j \in \mathbb{N}} \subset (0, \delta]$  with  $\lim_{j \rightarrow +\infty} t_{i_1, j} = 0$ . When  $i_2 \in \mathbb{N}$  satisfies  $(w_{i_2}, t_{i_2}) \neq (w_{i_2}, 0)$ , we take sequence  $\{t_{i_2, j}\}_{j \in \mathbb{N}} \subset (0, \delta]$  with  $\lim_{j \rightarrow +\infty} t_{i_2, j} = t_{i_2}$ . It follows from the argument in Case 2.1 and Case 2.2 that  $\lim_{i \rightarrow +\infty} \lim_{j \rightarrow +\infty} J(w_i, t_{i, j}) = \bar{J}(w, 0)$ .

Therefore  $\bar{J}$  is continuous. Inequalities (3) and (4) imply  $\bar{J}(\bar{D} \times (0, \delta]) \subset G$  and  $\bar{J}(\partial D \times \{0\}) \subset G$ . Moreover  $\bar{\pi} \circ \bar{J} = \chi$ . Now  $(G, \pi)$  is  $O_2$ -pseudoconvex

domain, therefore we obtain  $\overline{\overline{J}}(\overline{D} \times [0, \delta]) \subset G$ . However,

$$\begin{aligned} \chi(0, 0) &= (p(0), \dots, p(0), z_{k+1}^{(0)}, \dots, z_n^{(0)}) \\ &= (\exp(-h(0)), \dots, \exp(-h(0)), z_{k+1}^{(0)}, \dots, z_n^{(0)}) \\ &= (\exp(\log \delta_G(\pi_x^{-1}(0))), \dots, \exp(\log \delta_G(\pi_x^{-1}(0))), z_{k+1}^{(0)}, \dots, z_n^{(0)}) \\ &= (\delta_G(\pi_x^{-1}(0)), \dots, \delta_G(\pi_x^{-1}(0)), z_{k+1}^{(0)}, \dots, z_n^{(0)}) \\ &= \overline{\overline{\pi}}(u) \in \partial P^n(0). \end{aligned}$$

It follows that  $\overline{\overline{J}}(0, 0) = \psi_{\widehat{P}(0)} \circ (\overline{\overline{\pi}}_0)^{-1} \circ \chi(0, 0) = u \in \overline{\overline{\partial}}G$ . This is a contradiction. Therefore  $G$  is pseudoconvex.  $\square$

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